**CHAPTER THREE**

**THEORY OF QUADRATIC FUNCTIONS AND EQUATIONS**

Note the following important results in this topic:

If α and β are the roots of the quadratic equation ax2 + bx + c = 0 , then,

(1) Sum of the roots α + β = - $\frac{b}{a}$

(2) Product of the roots αβ = $\frac{c}{a}$

Note the following important result too:

(1) α2 + β2 = (α + β)2 - 2αβ

(2) α2 – β2 = (α + β)(α – β) = (α + β) (α + β)2 - 4αβ

(3) α - β = (α + β)2 - 4αβ

(4) α3 + β3 = (α + β)3 - 3αβ (α + β)

(5) α3 – β3 = (α - β)3 + 3αβ (α - β)

If you are given α and β with higher powers, rewrite them in terms of the suitable one above. For example:

α4 + β4 = (α2)2 + (β2)2 Let α2 be X and β2 be Y. So,

α4 + β4 = X2 +Y2 = (X + Y)2 – 2XY = (α2 + β2)2 - 2α2 β2 = {(α + β)2 - 2αβ}2 - 2(αβ)2

(6) To construct a quadratic equation whose roots are given (or known), use the formula:

x2 – (sum of roots)x + (product of roots) = 0

***QUESTION 1:***

(a) If α and β are roots of the equation x2 + px + q = 0, form a quadratic equation whose roots are (α2 + β2) and ( $\frac{1}{α }$ + $\frac{1}{ β} $).

(b) Find the values between which

 (x – 2)(x – 4)

 (x – 1)(x – 5) cannot lie for real values of x.

***Solution:***

 For x2 + px + q = 0, a = 1, b = p and c = q. α + β = - $\frac{b}{ a}$ = - $\frac{p}{ 1}$ = -p

αβ = $\frac{c}{ a}$ = $\frac{q}{ 1}$ = q

α2 + β2 = (α + β)2 - 2αβ = (-p)2 – 2q = (p2 – 2q)

$\frac{1}{α }$ + $\frac{1}{ β}$ = $\frac{β+α}{ αβ}$ = $\frac{α+β}{ αβ}$ = - $\frac{p}{ q}$

A quadraic equation whose roots are given is:

x2 – (sum of roots)x + (product of roots) = 0

Sum of roots = p2 – 2q – $\frac{p}{q}$ = p2q – 2q2 - p

 q

Products of roots = - $\frac{p}{q}$ (p2 – 2q) = - p3 + 2pq = -p3 + 2pq = 2pq - p3

 q q q q

$∴$ The required quadraic equation is:

 x2 – p2q – 2q2 - p x + 2pq – p3 = 0

 q q

px2 – (p2q – 2q – p)x + (2pq – p3) = 0

(b) Let y = (x – 2)(x – 4) .

 (x – 1)(x – 5)

y(x – 1)(x – 5) = (x – 2)(x – 4), y(x2 – 6x + 5) = x2 – 6x = 8,

yx2 – 6xy + 5y = x2 – 6x + 8, yx2 – x2 – 6xy + 6x + 5y – 8 = 0,

(y – 1)x2 + (6 – 6y)x + (5y – 8) = 0. For real values of x, b2 – 4ac ≥ 0 .

b = (6 – 6y), a = (y – 1), c = (5y – 8)

(6 – 6y)2 – 4(y – 1)(5y – 8) ≥ 0, 36 – 72y + 36y2 – 4(5y2 – 13y + 8) ≥ 0 ,

36 – 72y + 36y2 – 20y2 + 52y – 32 ≥ 0, 16y2 – 20y + 4 ≥ 0 ,

4y2 – 4y – 1y + 1 ≥ 0, 4y(y – 1) –1(y – 1) ≥ 0 ,

(y – 1)(4y – 1) ≥ 0. So, we have:

 $\frac{1}{4} $ 1

Hence, y lies in the range: y ≤ ¼ or y ≥ 1 .

Therefore, y cannot lie in the range: ¼ < y < 1

***QUESTION 2***

(a) Find the values of a for which the equation:

 (a + 3)x2 – (11a + 1)x + a = 2(a – 5) has equal roots.

(b) Find the range of values of λ for which both roots of the equation

 x2 – 6x – 1 + λ(2x + 1) = 0 are real and positive.

(c) Show that 2x2 + 3x + 7 is always positive for real values of x and find its

 minimum value.

(d) If the equation a2x2 + 6abx + ac + 8b2 = 0 has equal roots, prove that the roots of

 the equation ac(x + 1)2 = 4b2x are also equal.

***Solution:***

(a) (a + 3)x2 – (11a + 1)x + a = 2(a – 5)

 (a + 3)x2 – (11a + 1)x + (10 – a) = 0

For real and equal roots, the discriminant D = 0. So,

[-(11a + 1)]2 – 4(a + 3)(10 – a) = 0,

(11a + 1)(11a + 1) – 4(a + 3)(10 – a) = 0,

(121a2 + 22a + 1) – 4(10a – a2 + 30 – 3a) = 0,

(121a2 + 22a + 1) – 4(- a2 + 7a + 30) = 0,

(121a2 + 22a + 1) + 4a2 – 28a – 120 = 0,

125a2 – 6a – 119 = 0 . Solving using quadratic formula gives:

a = - (-6) ± (-6)2 – 4(125)(-199)

 2(125)

 a = 6 ± 36 + 56500 = 6 ± 244

 250 250

 = 6 + 244 or 6 – 244

 250 250

 $∴a$ = 1 or -0.952

(b) x2 – 6x – 1 + λ(2x + 1) = 0, x2 – 6x – 1 + 2λx + λ = 0 ,

 x2 + 2λx – 6x + λ – 1 = 0, x2 + (2λ – 6)x + (λ – 1) = 0 ,

For real values of x, the discriminant D ≥ 0. So,

(2λ – 6)2 – 4(1)(λ- 1) ≥ 0, (2λ – 6)(2λ – 6) – 4(λ – 1) ≥ 0,

4λ2 - 24λ + 36 – 4λ + 4 ≥ 0, 4λ2 - 28λ + 40 ≥ 0,

4(λ2 - 7λ + 10) ≥ 0, λ2 - 7λ + 10 ≥ 0 ,

λ2 - 5λ - 2λ + 10 ≥ 0, λ(λ – 5) –2(λ – 5) ≥ 0,

(λ – 5)(λ – 2) ≥ 0 . So, we have:

 2  5

 $∴$ λ ≤ 2 or λ ≥ 5

These are the two ranges of values of λ for which both roots of the equation are **real**.

For the given conditions: **real and positive**, we have to try a value each in λ ≤ 2 and x ≥ 5

in the equation x2 + (2λ – 6)x + (λ – 1) = 0 to determine which one is positive.

 So, for λ ≥ 5, let’s try 6:

x2 + [2(6) – 6]x + (6 – 1) = 0 x2 + 6x + 5 = 0.

With this, both roots would be negative. (You can verify).

So, for λ $\leq $ 2, let’s try 2:

x2 + [2(2) – 6]x + (2 – 1) = 0 x2 - 2x + 1 = 0.

With this, both roots would be positive.

Therefore, the range of values of λ for which both roots of the equation are real and positive

is: λ ≤ 2 .

Note: Always take note of the condition(s) attached to this type of question.

(c) Let y = 2x2 + 3x + 7. So, we are asked to show that y is always positive for real values of x. Applying the principle of completing the square, we have:

 y = 2[x2 + $\frac{3x}{2}$] + 7 = 2[(x + $\frac{3}{4}$)2 – ($\frac{3}{4}$)2] + 7

 = 2[(x + $\frac{3}{4}$)2 – $\frac{9}{16}$] + 7 = 2(x + $\frac{3}{4}$)2 – $\frac{9}{8}$ + 7

 $∴$ y = 2(x + $\frac{3}{4}$)2 + $\frac{47}{8}$

Since the squares of numbers are always positive, hence, 2x2 + 3x + 7 is positive.

From y = 2(x + $\frac{3}{4}$)2 + $\frac{47}{8}$ , its minimum value is: $\frac{47}{8}$ = 5$\frac{7}{8}$ .

Note: Alternative formula for finding the minimum or maximum value (turning point) of

 a quadratic equation is: y = 4ac – b2 , x = -b

 4a 2a

(d) If a2x2 + 6abx + ac + 8b2 = 0 has equal roots, D = 0. So,

 (6ab)2 – 4(a2)(ac + 8b2) = 0 36a2b2 – 4(a3c + 8a2b2) = 0

 4a2b2 – 4a3c = 0 b2 = ac ……………. (\*)

 Substituting this in ac(x + 1)2 = 4b2x, we have: ac(x + 1)2 = 4acx ,

(x + 1)2 = 4x, (x + 1)(x + 1) = 4x , x2 + 2x + 1 = 4x ,

x2 + 2x + 1 – 4x = 0, x2 – 2x + 1 = 0 , x2 – 1x – 1x + 1 = 0

x(x – 1) –1(x – 1) = 0, (x – 1)(x – 1) = 0

 $∴$ x = 1 (twice). (i.e equal roots)

***QUESTION 3***

Find the sum and product of the roots of the equation 2x2 + 3x – 2 = 0 .

***Solution:***

2x2 + 3x – 2 = 0 a = 2, b = 3 and c = -2.

So, sum of the roots, α + β = - $\frac{b}{a}$ = - $\frac{3}{2}$ . Product of the roots, αβ = $\frac{c}{a}$ = - $\frac{2}{2} $ = -1.

***QUESTION 4***

(a) For what values of k does the equation x2 + 9 = (4 + k)$x$ have real roots?

(b) If the roots of the equation x2 + 6A$x$ = 18A – 9 are 6$∝$ - 3 and 6β -3 ,

 (i) find the values of $∝$2 – β2

 (ii) Hence, show that if $∝$ = β, then, A = $\sqrt{2}$ – 1 or A = -( $\sqrt{2}$ + 1)

***Solution:***

(a) x2 + 9 = (4 + k)$x$ x2 – (4 + k)$x$ + 9 = 0

For real values of root $x$, D ≥ 0. So,

[-(4 + k)]2 – 4(1)(9) ≥ 0 , 16 + 8k + k2 – 36 ≥ 0 ,

k2 + 8k – 20 ≥ 0, k2 + 10k – 2k – 20 ≥ 0,

k(k + 10) – 2(k + 10) ≥ 0, (k + 10)(k – 2) ≥ 0

 -10 2

$∴$ k ≤ - 10 or k ≥ 2 for real values of $x$.

(b)(i) x2 + 6A$x$ = 18A – 9

 x2 + 6A$x$ + (9 – 18A) = 0 ………………. (i)

Sum of the roots = (6α – 3) + (6β – 3) = (6α + 6β – 6)

Product of the roots = (6α – 3)(6β – 3) = (36αβ - 18α - 18β + 9)

A quadratic equation with known roots is: $x$2 – (sum of roots)$x$ + (product of roots) = 0

So, the given equation can be rewritten as: x2 – (6α + 6β – 6)$x$ + (36αβ - 18α -18β + 9) = 0

Comparing the two: -(6α + 6β – 6) = 6A, 6 - 6α - 6β = 6A, 1 – α – β = A

$∴$ α + β = 1 – A

Also, 36αβ - 18α - 18β + 9 = 9 – 18A 18A = 18α + 18β - 36αβ,

A = (α+β) - 2αβ

2αβ = (α + β) - A 2αβ = (1 – A) – A, 2αβ = 1 – 2A, $∴$ αβ = $\frac{1-2A}{2}$

α2 – β2 = (α + β)(α – β) = (α + β) (α + β)2 - 4αβ

Subsistituting for (α + β) and αβ, we have: α2 – β2 = (1 – A) (1 - A)2 – 4(1 - 2A)

 2

= (1 – A) 1 - 2A + A2 – 2 + 4A

= (1 – A) A2 + 2A – 1 $∴$ α2 – β2 = (1 – A) A2 + 2A – 1

(ii) If α = β, then, β2 - β2 = (1 - A) A2 + 2A – 1 = 0

 A2 + 2A – 1 = 0 A2 + 2A – 1 = 0 . Solving using quadratic formula gives:

A = -2 ± 22 – 4(1)(-1) = -2 ± $\sqrt{4+4}$

 2(1) 2

= -2 ± $\sqrt{8}$ = -2 ± 2$\sqrt{2}$ = -2 + 2$\sqrt{2}$ or -2 - 2$\sqrt{2}$

 2 2 2 2

$∴$ A = -1 + $\sqrt{2}$ or -1 – $\sqrt{2}$

***QUESTION 5***

1. Determine the values between which λ must not lie in order that the equation

 x2 + (5 + λ)$x$ + (λ + 5) = 0 has real roots.

1. Find the values of λ for which one root is 3 times the other, and
2. find the roots corresponding to each value of λ

***Solution:***

For x2 + (5 + λ)$x$ + (λ + 5) = 0 , a = 1, b = (5 + λ) and c = (λ + 5).

For real values of $x$, the discriminant b2 – 4ac ≥ 0 . So,

 (5 + λ)2 – 4(1)(λ + 5) ≥ 0, (5 + λ) (5 + λ) – 4(λ + 5) ≥ 0,

25 + 10λ + λ2 - 4λ – 20 ≥ 0, λ2 + 6λ + 5 ≥ 0,

λ2 + 5λ + 1λ + 5 ≥ 0, λ(λ + 5) + 1(λ + 5) ≥ 0

(λ + 5)(λ + 1) ≥ 0

 -5 -1

 λ = ≤ -5 or λ ≥ -1

Hence, for the equation to have real roots, λ must lie in the range: λ = ≤ -5 or λ ≥ -1

Conversely, for the equation to have real roots, λ must not lie range: -5 < λ -1

(ii) If one root is 3 times the other, let’s have the roots to be α and 3α. So,

sum of roots = α + 3α = 4α

product of roots = 3α2

The quadratic equation $x$2 + (5 + λ)$x$ + (λ + 5) = 0 can be written as:

$x$2 - 4α$x$ + 3α2 = 0 .

Comparing the coefficients:

-4α = 5 + λ λ + 4α = - 5 ………………(i)

3α2 = λ + 5 λ - 3α2 = - 5 ……………….(ii)

Solving (i) and (ii) simultaneously:

λ + 4α = - 5 ………………(i)

λ - 3α2 = - 5 ……………….(ii)

 4α + 3α2 = 0 α(4 + 3α) = 0 $∴$ α = 0 or - $\frac{4}{3}$

Substitute these in (i): λ + 4(0) = - 5 or λ + 4(-$\frac{4}{3}$) = - 5

λ = - 5 or λ = - 5 + $\frac{16}{3}$ = $\frac{1}{3}$

 $∴$ The required λ values are: - 5 and $\frac{1}{3}$

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