# CHAPTER 6
# MATRICES AND DETERMINANTS

## QUESTION 1

(a) The equation $M^2 = aM + bI$, where a, b are real scalars, is satisfied by the matrix M given by:

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

(i) Find the values of a and b.
(ii) Use the equation to find the inverse of the matrix M.
(iii) Hence, solve the following equations:

$x + 2y + 2z = 3$
$2x + y + 2z = 1$
$2x + 2y + z = 1$

(b) Let $A = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

Compute $f(A) = A^3 - 3A^2 + 2A + I$ .

***Solution***:

(a) (i) $M^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} =$

$$\begin{pmatrix} (1x1)+(2x2)+(2x2) & (1x2)+(2x1)+(2x2) & (1x1)+(2x2)+(2x1) \\ (2x1)+(1x2)+(2x2) & (2x2)+(1x1)+(2x2) & (2x2)+(1x2)+(2x1) \\ (2x1)+(2x2)+(1x2) & (2x2)+(2x1)+(1x2) & (2x2)+(2x2)+(1x1) \end{pmatrix} =$$

$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$.   So, $M^2 = aM + bI$ ⟶

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}^2 = a\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remember I is an Identity Matrix. ⟶

$\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} = \begin{pmatrix} a & 2a & 2a \\ 2a & a & 2a \\ 2a & 2a & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$. We can rewrite this equation as:

$$\begin{pmatrix} a & 2a & 2a \\ 2a & a & 2a \\ 2a & 2a & a \end{pmatrix} + \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

So, from row 1, $a + b = 9$ ———(i)   $2a + 0 = 8$ ———(ii)

⟶ $a = 4$.   So, (i): $4 + b = 9$   $\therefore b = 5$.

(ii) Remember,

Inverse of matrix X The matrix = Identity matrix

So, for the matrix M, $M^{-1} \times M = I$ ⟶

$M^{-1} = \frac{I}{M}$. Now, we try to establish $\frac{I}{M}$ from the given equation:

$M^2 = aM + bI$. Dividing through by aM ⟶

$\frac{M^2}{aM} = \frac{aM}{aM} + \frac{bI}{aM}$ ⟶ $\frac{M}{a} = 1 + \frac{bI}{aM}$, ⟶

$\frac{bI}{aM} = \frac{M}{a} - 1$ $\therefore \frac{I}{M} = \left[\frac{M}{a} - 1\right] \times \frac{a}{b} = \frac{aM}{ab} - \frac{a}{b} = \frac{M}{b} - \frac{a}{b} = \frac{M-a}{b}$

Inverse of matrix M i.e $\frac{I}{M} = \frac{I}{b}(M - a)$

$$= \frac{1}{5}\left(\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 4\right) = \frac{1}{5}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - \frac{4}{5}$$

$$= \begin{pmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

( Note the introduction of Identity matrix I adjacent to $\frac{4}{5}$. This is because we can subtract only a matrix from another matrix.)

$$= \begin{pmatrix} 1/5 & 2/5 & 2/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & 2/5 & 1/5 \end{pmatrix} - \begin{pmatrix} 4/5 & 0 & 0 \\ 0 & 4/5 & 0 \\ 0 & 0 & 4/5 \end{pmatrix} = \begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$$

$\therefore$ The inverse of matrix M (i.e $M^{-1}$) = $\begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}$.

(iii) $x + 2y + 2z = 3$
$2x + y + 2z = 1$ (Given)
$2x + 2y + z = 1$

Rewriting these equations in matrix form gives:

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Remember, $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = M$. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be A and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ be .

So, we now have:

MA = B ⟶ a = $M^{-1}B$ (Note that $M^{-1}$ must come before B i.e $M^{-1}B$ not $BM^{-1}$)

$$\longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -9/5 & 2/5 & 2/5 \\ 6/5 & -3/5 & 2/5 \\ 6/5 & 2/5 & -3/5 \end{pmatrix} \qquad \therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ x = -1, y = 1 and z = 1.

(b) (i) $A = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ $\qquad A^2 = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 14 & 15 \\ 11 & 13 & 13 \\ 5 & 6 & 6 \end{pmatrix}$

$$A^3 = \begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 14 & 15 \\ 11 & 13 & 13 \\ 5 & 6 & 6 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 67 & 79 & 81 \\ 61 & 72 & 74 \\ 28 & 33 & 34 \end{pmatrix}$$

Hence, $f(A) = A^3 - 3A^2 + 2A + I \longrightarrow$

$$= \begin{pmatrix} 67 & 79 & 81 \\ 61 & 72 & 74 \\ 28 & 33 & 34 \end{pmatrix} - 3\begin{pmatrix} 12 & 14 & 15 \\ 11 & 13 & 13 \\ 5 & 6 & 6 \end{pmatrix} + 2\begin{pmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 67 & 79 & 81 \\ 61 & 72 & 74 \\ 28 & 33 & 34 \end{pmatrix} - \begin{pmatrix} 36 & 42 & 45 \\ 33 & 39 & 39 \\ 15 & 18 & 18 \end{pmatrix} + \begin{pmatrix} 4 & 6 & 4 \\ 4 & 4 & 6 \\ 2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 36 & 43 & 40 \\ 32 & 38 & 41 \\ 15 & 17 & 19 \end{pmatrix}.$$

## QUESTION 2

(a) (i) Find the inverse of the matrix $P = \begin{pmatrix} 1 & 2 & k \\ 2 & k & 1 \\ k & 1 & 2 \end{pmatrix}$ where k is any real number.

(ii) Hence or otherwise, solve the following system of equations:

$x + 2y = 4 - 3z$
$2x + 3y + z = 0$
$3x + y + 2z + 10 = 0$

(b) Let $g(x) = x^3 + 3x^2 - 2$. Find $g(A)$, where A is the matrix $\begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

**Solution:**

(i) Remember, Inverse of matrix P is: $P^{-1} = \dfrac{Adj.P}{|P|}$

Now, expanding along the first row for P:

$$|P| = 1\begin{vmatrix} k & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ k & 2 \end{vmatrix} + k\begin{vmatrix} 2 & k \\ k & 1 \end{vmatrix} = 1(2k - 1) - 2(4 - k) + k(2 - k^2)$$

$= 2k - 1 - 8 + 2k + 2k - k^3 = 6k - k^3 - 9$ . $\therefore$ $|P| = 6k - k^3 - 9$ .

For the Adj.P, let the matrix of cofactors be:

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

$C_{11} = +\begin{vmatrix} k & 1 \\ 1 & 2 \end{vmatrix} = 2k - 1$, $C_{12} = -\begin{vmatrix} 2 & 1 \\ k & 2 \end{vmatrix} = -(4 - k) = k - 4$, $C_{13} = +\begin{vmatrix} 2 & k \\ k & 1 \end{vmatrix} = 2 - k^2$

$C_{21} = -\begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = -(4 - k) = k - 4$, $C_{22} = +\begin{vmatrix} 1 & k \\ k & 2 \end{vmatrix} = 2 - k^2$, $C_{23} = -\begin{vmatrix} 1 & 2 \\ k & 1 \end{vmatrix} = -(1 - 2k) = 2k - 1$,

$C_{31} = +\begin{vmatrix} 2 & k \\ k & 1 \end{vmatrix} = 2 - k^2$, $C_{32} = -\begin{vmatrix} 1 & k \\ 2 & 1 \end{vmatrix} = -(1 - 2k) = 2k - 1$, $C_{33} = +\begin{vmatrix} 1 & 2 \\ 2 & k \end{vmatrix} = k - 4$ .

Take note of how the sign changes for cofactors i.e $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

So, the matrix of cofactors is now:

$$C = \begin{pmatrix} (2k - 1) & (k - 4) & (2 - k^2) \\ (k - 4) & (2 - k^2) & (2k - 1) \\ (2 - k^2) & (2k - 1) & (k - 4) \end{pmatrix}.$$

Transpose of this gives the Adj.P $= C^{-1} = \begin{pmatrix} (2k - 1) & (k - 4) & (2 - k^2) \\ (k - 4) & (2 - k^2) & (2k - 1) \\ (2 - k^2) & (2k - 1) & (k - 4) \end{pmatrix}$.

[The same thing i.e $C^{-1} = C$ ( Symmetric)] .

$\therefore$ The inverse of the given matrix is:

$$P^{-1} = \frac{Adj.P}{P} = \frac{1}{6k - k^3 - 9}\begin{pmatrix} (2k - 1) & (k - 4) & (2 - k^2) \\ (k - 4) & (2 - k^2) & (2k - 1) \\ (2 - k^2) & (2k - 1) & (k - 4) \end{pmatrix}$$

(ii) $x + 2y = 4 - 3z$ → $x + 2y + 3z = 4$

$2x + 3y + z = 0$ → $2x + 3y + z = 0$

$3x + y + 2z + 10 = 0$ → $3x + y + 2z = -10$

Rewriting these equations in matrix form gives:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}$$

When you compare $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ with $\begin{pmatrix} 1 & 2 & k \\ 2 & k & 1 \\ k & 1 & 2 \end{pmatrix}$, you would find that k = 3.

When you now substitute k = 3 into $\frac{1}{6k - k^3 - 9}\begin{pmatrix} (2k-1) & (k-4) & (2-k^2) \\ (k-4) & (2-k^2) & (2k-1) \\ (2-k^2) & (2k-1) & (k-4) \end{pmatrix}$, that will give you

the inverse of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$.

So, inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} = -\frac{1}{18}\begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix} = \begin{pmatrix} -5/18 & 1/18 & 7/18 \\ 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \end{pmatrix}$.

Now, back to the equation:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}$$

Let $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ be P, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be Q and $\begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}$ be R. ⟶

PQ = R , ∴ $Q = P^{-1}R$ where $P^{-1}$ is the inverse.
(Note that $P^{-1}$ must come before R i.e $P^{-1}R$ not $RP^{-1}$.)

So, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5/18 & 1/18 & 7/18 \\ 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix}$ ⟶

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -{}^{20}/_{18} + 0 - {}^{70}/_{18} \\ {}^{4}/_{18} + 0 + {}^{50}/_{18} \\ {}^{28}/_{18} + 0 - {}^{10}/_{18} \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$$ ∴ $x = -5$, $y = 3$ and $z = 1$.

(b) If $g(x) = x^3 + 3x^2 - 2$, then $g(A) = A^3 + 3A - 2$ .

$$A = \begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (-2 \times -2) + (1\times2) + (-3\times1) & (-2\times1) + (1\times3) + (-3 \times -1) & (-2 \times -3) + (1\times1) + (-3 \times 0) \\ (2 \times -2) + (3\times2) + (1\times1) & (2\times1) + (3\times3) + (1 \times -1) & (2 \times -3) + (3 \times 1) + (1 \times 0) \\ (1 \times -2) + (-1\times2) + (0\times1) & (1\times1) + (-1 \times 3) + (0 \times -1) & (1 \times -3) + (-1 \times 1) + (0\times0) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 7 \\ 3 & 10 & -3 \\ -4 & -2 & -4 \end{pmatrix}.$$

$$A^3 = \begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & 7 \\ 3 & 10 & -3 \\ -4 & -2 & -4 \end{pmatrix}\begin{pmatrix} -2 & 1 & -3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 8 & -5 \\ 11 & 36 & 1 \\ 0 & -6 & 10 \end{pmatrix}.$$

Hence, $g(A) = A^3 + 3A - 2$

$$= \begin{pmatrix} 9 & 8 & -5 \\ 11 & 36 & -1 \\ 0 & -6 & 10 \end{pmatrix} + 3\begin{pmatrix} 3 & 4 & 7 \\ 3 & 1 & -3 \\ -4 & -2 & -4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & -5 \\ 11 & 36 & -1 \\ 0 & -6 & 10 \end{pmatrix} + \begin{pmatrix} 9 & 12 & 21 \\ 9 & 30 & -9 \\ -12 & -6 & -12 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 16 & 20 & 16 \\ 20 & 64 & -10 \\ -12 & -12 & 0 \end{pmatrix}.$$

**<u>QUESTION 3</u>**

(a) Given that $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b2 \\ 1 & c & c^2 \end{pmatrix}$, find:

(i) $A^{-1}$ (ii) $ABA^{-1}$ . (iii) Show that $|ABA^{-1}| = |B|$.

(b) Using Cramer's Rule only, solve the following system of equations:

$$2x - 3y + z = 2$$
$$x + y + z = 1$$
$$2x - 2y + z = 3$$

***<u>Solution:</u>***

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