CHAPTER 5
THE BINOMIAL THEOREM

QUESTION 1

(a) Prove by Mathematical induction that
$2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = 2n^2(n + 1)^2$.

(b) Find the first FIVE terms in the expansion of $(1 + x)^4 (1 - 2x)^5$ in ascending powers of x.

(c) Use the expansion of $(1 + x)^6$ to find the value of 1.98^6 correct to 5 decimal places.

Solution:

(a) $2^3, 4^3, 6^3, \ldots , (2n)^3$ is a sequence with the general formula $(2n)^3$ for every term (i.e. general term).

$2^3 + 4^3 + 6^3 + \ldots + (2n)^3$ is the sum of the terms in the sequence from the first term to the n^{th} term. In other words, $2^3 + 4^3 + 6^3 + \ldots + (2n)^3 = {}^n\Sigma_1 (2n)^3$. The first term is 2^3. The second term is 4^3. The third term is 6^3. The n^{th} term is $(2n)^3$ (As shown in the sequence). So, we are asked to prove that the sum of the first n terms, ${}^n\Sigma_1 (2n)^3 = 2n^2(n + 1)^2$.

So, when n = 1, LHS = 2^3;

RHS = $2(1)^2 (1 + 1)^2 = 2^3$

$\longrightarrow$ The formula is true for n = 1

When n = r, $r \epsilon N$, we have:

$2^3 + 4^3 + 6^3 + \ldots + (2r)^3 = 2r^2(r + 1)^2$ (i.e the sum of the terms from the 1^{st} term to the r^{th} term).

Hence, when n = r + 1, we have:

$2^3 + 4^3 + 6^3 + \ldots + (2r)^3 + [2(r + 1)]^3 = 2r^2(r + 1)^2 + [2(r + 1)]^3$ (i.e the sum of the terms from the 1^{st} term to the $(r + 1)^{th}$ term). $= 2r^2 (r + 1)^2 + 2^3(r + 1)^3 = 2r^2 (r + 1)^2 + 8 (r + 1)^3$

$= 2(r + 1)^2 [r^2 + 4(r + 1)] = 2(r + 1)^2 (r^2 + 4r + 4)$

$= 2(r + 1)^2 (r + 2)^2 = 2(r + 1)^2 [(r + 1) + 1]^2$

$\therefore$ The formula is true for all positive integers. If you observe the answer we got for n = r + 1, i.e $2(r + 1)^2 [r + 1) + 1]^2$, you will discover that it is similar to the answer given in the equation i.e $2n^2 (n + 1)^2$

(b) Let's use expansion with Pasca'sl Triangle:

Power (n)	Coefficients [for $(x + y)^n$]
0 $\longrightarrow$	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

6 1 6 15 20 15 6 1

$(1 + x)^4 = 1(1)^4x^0 + 4(1)^3x^1 + 6(1)^2x^2 + 4(1)^1x^3 + 1(1)^0x^4$
$= 1 + 4x + 6x^2 + 4x^3 + x^4$

$(1 – 2x)^5 = [1 + (-2x)]^5$.

So, $[1 + (-2x)]^5 = 1(1)^5(-2x)^0 + 5(1)^4(-2x)^1 + 10(1)^3(-2x)^2 + 10(1)^2(-2x)^3 + 5(1)^1(-2x)^4 + 1(1)^0(-2x)^5$
$= 1 – 10x + 40x^2 – 80x^3 + 80x^4 – 32x^5$

Hence,
$(1 + x)^4 (1 – 2x)^5 = (1 + 4x + 6x^2 + 4x^3 + x^4)(1 – 10x + 40x^2 – 80x^3 + 80x^4 – 32x^5)$

Remember that we are asked to find the first FIVE terms in the expansion of $(1 + x)^4 (1 – 2x)^5$. You don't need to expand all in order to get the first FIVE terms. Expanding all is a waste of time. What then do we do?
Now, be attentive.
The highest power of x in $(1 + x)^4$ is 4 and the highest power of x in $(1 – 2x)^5$ is 5 as shown in their expanded forms. So, if $(1 – x)^4 (1 – 2x)^5$ is fully expanded and all the like terms are collected together and the expansion is arranged in ascending order of x, the highest power of x in the expansion would
be 9 (i.e 4 + 5) and the least power x would, of course, be zero ($x^0 = 1$).
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So, we are going to have 10 terms in the expansion. It is obvious that the first term in the expansion would be 1. The first FIVE terms would be from 1 to the term with x^4.
So, we need to expand $(1 + 4x + 6x^2 + 4x^3 + x^4)(1 – 10x + 40x^2 – 80x^3 – 32x^5)$ up to x^4 and neglect higher power. So, let's do it:
$(1 + 4x + 6x^2 + 4x^3 + x^4)(1 –10x + 40x^2 –80x^3 + 80x^4 – 32x^5)$
$= 1 – 10x + 40x^2 – 80x^3 + 80x^4 + 4x – 40x^2 + 160x^3 – 320x^4 + 6x^2 - 60x^3 + 240x^4 + 4x^3 – 40x^4 + x^4$
$= 1 – 6x + 6x^2 + 24x^3 – 39x^4$

(c) Using Pascal's Triangle, we have:
$(1 + x)^6 = 1(1)^6x^0 + 6(1)^5x^1 + 15(1)^4x^2 + 20(1)^3x^3 + 15(1)^2x^4 + 6(1)^1x^5 + 1(1)^0x^6$
$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

$1.98 = 2 – 0.02 = 2(1 – 0.01)$
So, $1.98^6 = [2(1 – 0.01)]^6 = 2^6 (1 – 0.01)^6$
$(1 – 0.01)^6 = [1 + (-0.01)]^6$
Comparing $(1 + x)^6$ with $[1 + (-0.01)]^6$, $x = -0.01$
So, $[1 + (-0.06)]^6 = 1 + 6(-0.01) + 15(-0.01)^2 + 20(-0.01)^3 + 15(-0.01)^4 + 6(-0.01)^5 + (-0.01)^6$
$= 1 – 0.06 + 0.0015 – (2 \times 10^{-5}) + (1.5 \times 10^{-7}) – (6 \times 10^{-10}) + (1 \times 10^{-12})$
$= 0.941480149$

Remember $1.98^6 = 2^6 (1 - 0.01)^6$
$\therefore\ 1.98^6 = 2^6 \times 0.941480149 = 60.25473$ (5 d.p.)

QUESTION 2

(a) Find the term independent of x in the expansion

$$\left(\frac{4y^3}{x^4} - \frac{x^2}{2y^2}\right)^{12}$$

(b) By expanding $(1 - 0.125)^{1/3}$ by the binomial theorem, calculate $\sqrt[3]{7}$ to five significant figures.

(c) Prove by induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1)$$

Solution:

(a) Remember that binomial theorem formula is:

$(a + b)^n = {}^nC_0\, a^{n-0}\, b^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + {}^nC_4\, a^{n-4}\, b^4 + \ldots + \underline{{}^nC_r\, a^{n-r}\, b^r} + \ldots + {}^nC_n\, a^{n-n}\, b^n \longrightarrow$

$(a + b)^n = a^n + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + {}^nC_4\, a^{n-4}\, b^4 + \ldots + \underline{{}^nC_r\, a^{n-r}\, b^r} + \ldots\, b^n$

The term underlined i.e ${}^nC_r\, a^{n-r}\, b^r$ is the <u>general term.</u> That is, it is the formula for every term in the expansion. The relationship between the n^{th} term and r in the expansion is: $n^{th} = r + 1$ i.e

for 3^{rd} term, $3 = r + 1 \longrightarrow r = 2.$
For the first term, $1 = r + 1 \longrightarrow r = 0$.

If you are asked to find any term in the expansion of a binomial expression with higher power [e.g $(a + b)^{30}$, $(a + b)^{12}$ etc.], you don't need to expand all before getting the term. Expanding all is a waste of time. All you need to do is to use the general term ${}^nC_r\, a^{n-r}\, b^r$. Let's use the following question to illustrate:
Find the fourth term in the expansion $(a + b)^5$.

<u>Solution:</u> The General term is: ${}^nC_r\, a^{n-r}\, b^r$. Remember we are told to find the fourth term. So, using
$n^{th} = r + 1, \quad 4 = r + 1 \longrightarrow r = 3.$
The power n = 5. So, the fourth term is now: ${}^5C_3\, a^{5-3}\, b^3 = {}^5C_3\, a^2\, b^3$

$${}^5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!} = \frac{5 \times 4 \times 3!}{2!3!} = 10$$

$\therefore\ {}^5C_3\, a^2\, b^3 = 10a^2b^3$
$\therefore$ The fourth term in the expansion $(a + b)^5$ is $10a^2b^3$. You may confirm this answer by fully expanding $(a + b)^5$ and find its fourth term.

Now, to the main question: We are asked to find the term independent of x in the expansion

$$\left(\frac{4y^3}{x^4} - \frac{x^2}{2y^2}\right)^{12}$$

Solution:

$$\left(\frac{4y^3}{x^4} - \frac{x^2}{2y^2}\right)^{12} = \left(\frac{4y^3}{x^4} + \left(\frac{-x^2}{2y^2}\right)\right)^{12}$$

Binomial theorem considers "plus" (+) to be between the two terms. If "minus" (-) is present between the two terms, you have to adjust as shown above.

Comparing the expression with $(a + b)^n \longrightarrow a = \frac{4y^3}{x^4}$, $b = -\frac{x^2}{2y^2}$, $n = 12$

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So, the general term is:

$${}^nC_r\, a^{n-r} b^r = {}^{12}C_r \left(\frac{4y^3}{x^4}\right)^{12-r} \left(\frac{-x^2}{2y^2}\right)^r$$

$$= {}^{12}C_r \left[\frac{4^{(12-r)}\, y^{3(12-r)}}{x^{4(12-r)}}\right]\left[\frac{(-1)(x^2)}{2y^2}\right]^r$$

$$= {}^{12}C_r \left[\frac{4^{(12-r)}\, y^{(36-3r)}}{x^{(48-4r)}}\right]\left[\frac{(-1)^r\, x^{2r}}{2^r y^{2r}}\right] = {}^{12}C_r \left[\frac{4^{(12-r)} y^{(36-3r)}\, (-1)^r}{2^r\, y^{2r}}\right]\left[\frac{x^{2r}}{x^{(48-4r)}}\right]$$

$$= {}^{12}C_r \left[\frac{(-1)^r\, 4^{(12-r)}\, y^{(36-3r)}}{2^r\, y^{2r}}\right]\left[x^{2r} . x^{-(48-4r)}\right]$$

$$= {}^{12}C_r \left[\frac{(-1)^r\, 4^{(12-r)}\, y^{(36-3r)}}{2^r\, y^{2r}}\right]\left[x^{2r} . x^{(4r-48)}\right]$$

$$= {}^{12}C_r \left[\frac{(-1)^r\, 4^{(12-r)}\, y^{(36-3r)}}{2^r\, y^{2r}}\right]\left[x^{(2r+4r-48)}\right]$$

$$= {}^{12}C_r \left[\frac{(-1)^r\, 4^{(12-r)}\, y^{(36-3r)}}{2^r\, y^{2r}}\right]\left[x^{(6r-48)}\right]$$

(What is mainly done here is the collection of all x terms together i.e x^{6r-48}).

Note, "the term independent of x in the expansion" is the term that does not contain x. It is the term with x^0. (Note that $x^0 = 1$).

Hence,

$x^{6r-48} = x^0 \longrightarrow 6r - 48 = 0 \longrightarrow 6r = 48 \quad \therefore r = 8$

Remember that $n^{th} = r + 1$. So, $n^{th} = 8 + 1 \quad \therefore n = 9$

$\therefore$ The term independent of x in the expansion is the 9^{th} term which is equal to

$$^{12}C_r \left(\frac{(-1)^8\, 4^{(12-8)}\, y^{(36-4)}}{2^8\, y^{16}}\right) \qquad \text{(Remember } r = 8) \longrightarrow$$

$$= {}^{12}C_8 \left(\frac{(1)\, 4^4\, y^{12}}{2^8\, y^{16}}\right) = {}^{12}C_8 \left(\frac{256\, y^{12}}{256\, y^{16}}\right) = {}^{12}C_8\, y^{-4}$$

$$^{12}C_8 = \frac{12!}{(12-8)!8!} = \frac{12!}{4!8!} = \frac{12 \times 11 \times 10 \times 9 \times 8!}{4!8!}$$

$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495 \qquad \therefore\ {}^{12}C_8\, y^{-4} = 495y^{-4} = \frac{495}{y^4}$$

$\therefore$ The term independent of x in the expansion is $= \frac{495}{y^4}$ (9^{th} term)

(b) Remember

$(a + b)^n = a^n + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}\, b^2 + {}^nC_3\, a^{n-3}\, b^3 + {}^nC_4\, a^{n-4}\, b^4\, n + \ldots + {}^nC_r\, a^{b-r}\, b^r + \ldots \cancel{+ b^n}$

(i)

(Binomial Expansion Formula)

This can be rewritten as:

$(a + b)^n = a^n + n\, a^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \ldots + bn$

(ii)

This is because: ${}^nC_1 = n$, ${}^nC_2 = \frac{n(n-1)}{2!}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$,

${}^nC_4 = \frac{n(n-1)(n-2)(n-3)}{4!}$, etc.

For the expansion of binomial expression with **negative** and **fraction powers**, use:

$(1 + x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$

The condition attached to this is $-1 < x < 1$ (i.e $|x| < 1$)

<u>Note this</u>: In the expansion $(1 + x)^n$, when n is a positive integer, the expansion terminates with x^n (i.e x^n will be the last term). When n is not a positive integer, the expansion will not terminate with x^n and the condition $-1 < x < 1$ is absolutely essential.

If the binomial expression is given in the form other than $(1 + x)^n$, rewrite (unless otherwise stated) them in terms of $(1 + x)^n$. For example, given $(2 + x)^{-1}$, rewrite as $[2(1 + \frac{x}{2})]^{-1} = 2^{-1}(1 + \frac{x}{2})^{-1}$. You can now expand $(1 + \frac{x}{2})^{-1}$ first using the above formula. In this case, the condition attached would be

$-1 < \frac{x}{2} < 1$ (i.e $\left|\frac{x}{2}\right| < 1$).

Note that you cannot use Pascal's Triangle in the expansion of binomial expression with negative and fractional powers. This is one of the advantages binomial expansion formula has over the expansion using Pascal triangle.

Now, to the question. We are asked: "By expanding $(1 - 0.125)^{1/3}$ by binomial theorem, calculate $\sqrt[3]{7}$ to five significant figures".

Solution:

$(1 - 0.125)^{1/3} = [1 + (-0.125)]^{1/3}$

Comparing $(1 + x)^n$ with $[1 + (-0.125)]^{1/3}$ $\longrightarrow$ $x = -0.125$, $n = {}^1/_3$.

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

$$[1 + (-0.125)]^{1/3} = 1 + ({}^1/_3)(-0.125) + \frac{({}^1/_3)(-{}^2/_3)}{2!}(-0.125)^2 + \frac{({}^1/_3)(-{}^2/_3)(-{}^5/_3)\,(-0.125)^3}{3!}$$

$= 1 - 0.04167 - 0.00174 - 0.00012$

$= 0.95647$

Now,

$(1 - 0.125)^{1/3} = 0.95647$ $\qquad \sqrt[3]{7} = 7^{1/3}$

Comparing $(1 - 0.125)^{1/3}$ with $7^{1/3}$, you will observe that there is similarity between them. All you need to do now is to write 7 in terms of $(1 - 0.125)$ So, we have:

$7 = 8 - 1 = 8(1 - 0.125)$. Hence,

$7^{1/3} = [8(1 - 0.125)]^{1/3} = 8^{1/3}(1 - 0.125)^{1/3} = 2(0.95647) = 1.91294$

$\sqrt[3]{7} = 1.9129$ (to 5 s.f)

(c) The question here is similar to question 1a above. We are told

"Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$"

Solution:

Note that $1^2 + 2^2 + 3^2 + \ldots + n^2 \longrightarrow {}^n\Sigma_1 n^2$ i.e the sum of the first nth terms of the series.

So, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1) \longrightarrow$

${}^n\Sigma_1\, n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

When n = 1, LHS = 1^2, RHS = $\frac{1}{6}(1)(1+1)(2+1) = \frac{1}{6}(2)(3) = 1^2$ $\longrightarrow$

The formula is true for n = 1 .

When n = r, r ϵ N, we have:

$1^2 + 2^2 + 3^2 + \ldots + r^2 = \frac{1}{6}r(r + 1)(2r + 1)$ [i.e the sum of the terms from the 1st term to the rth term].

Hence, when n = r + 1, we have:

$1^2 + 2^2 + 3^2 + \ldots + r^2 + (r + 1)^2 = \frac{1}{6}r(r + 1)(2r + 1) + (r + 1)^2$ [i.e the sum of the terms from the

1st term to the $(r + 1)^{th}$ term and note that $1^2 + 2^2 + 3^2 + \ldots + r^2 = \frac{1}{6}r(r + 1)(2r + 1)$] .

$= (r + 1)\,[\frac{1}{6}r(2r + 1) + (r + 1)] = (r + 1)[\frac{1}{6}(2r^2 + r) + (r + 1)]$

$= (r + 1)\,[\frac{r2}{3} + \frac{r}{6} + r + 1] = (r + 1)\left(\frac{2r^2 + r + 6r + 6}{6}\right)$

$= (r + 1)\,[\frac{1}{6}(2r^2 + 7r + 6)] = \frac{1}{6}(r + 1)\,[(r + 2)(2r + 3)]$

$= \frac{1}{6}(r + 1)(r + 2)(2r + 3) = \frac{1}{6}(r + 1)\,[(r + 1) + 1]\,[2r + 2) + 1]$

$= \frac{1}{6}(r + 1)\,[(r + 1) + 1][2(r + 1) + 1]$

The formula is true for all positive integers. If you observe the answer we got for n = r + 1 i.e = $\frac{1}{6}(r + 1)[(r + 1) + 1][2(r + 1) + 1]$, you will discover that it is similar to the answer given in the equation i.e $\frac{1}{6}n(n + 1)(2n + 1)$.

Note: If you are asked to prove any formula using Mathematical Induction, all you need to do is to prove that the formula is true for **three** positive integers. The positive integers are **n = 1**, and any two consecutive positiveve integers **n = r** and **n = r + 1** .

To get the complete past questions and solutions/explanations on **Binomial Theorem**, you can contact: 08033487161, 08177093682 or osospecial2015@yahoo.com for just N500 ($1). You can also get the past questions and solutions/explanations for the remaining topics on MTH 101.